Stress and Strain Tensors

Stress at a point.

Imagine an arbitrary solid body oriented in a cartesian coordinate system. A number of forces are acting on this body in different directions but the net force (the vector sum of the forces) on the body is 0. Conceptually slice the body on a plane normal to the x-direction (parallel to the yz-plane). Take a small area on this plane and call it $\Delta\mathbf{A}_x = \Delta A_x \hat{\mathbf{x}}$. Calculate the resolved force acting on this small area and call it $\Delta\mathbf{F}$.

$$\Delta\mathbf{F} = \Delta\mathbf{F}_x + \Delta\mathbf{F}_y + \Delta\mathbf{F}_z = \Delta F_x \hat{\mathbf{x}} + \Delta F_y \hat{\mathbf{y}} + \Delta F_z \hat{\mathbf{z}}.$$

Notice that since $\Delta\mathbf{F}$ is the *total* force acting *only* on $\Delta\mathbf{A}_x$, the magnitude of $\Delta\mathbf{F}$ will change as $\Delta\mathbf{A}_x$ changes.

We can define three scalar quantities,

$$\sigma_{xx} = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x},\ \sigma_{xy} = \tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x},\ \text{and}\ \sigma_{xz} = \tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x}.$$

The first subscript refers to the plane and the second refers to the force direction. If we do the same conceptual experiment at the same location but in the y and z-directions, we obtain

$$\sigma_{yy} = \lim_{\Delta A_y \to 0} \frac{\Delta F_y}{\Delta A_y},\ \sigma_{yx} = \tau_{yx} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x}{\Delta A_y},\ \sigma_{yz} = \tau_{yz} = \lim_{\Delta A_y \to 0} \frac{\Delta F_z}{\Delta A_y},$$

$$\sigma_{zz} = \lim_{\Delta A_z \to 0} \frac{\Delta F_z}{\Delta A_z},\ \sigma_{zx} = \tau_{zx} = \lim_{\Delta A_z \to 0} \frac{\Delta F_x}{\Delta A_z},\ \text{and}\ \sigma_{zy} = \tau_{zy} = \lim_{\Delta A_z \to 0} \frac{\Delta F_y}{\Delta A_z}.$$

For static equilibrium $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{xz} = \sigma_{zx}$, resulting in six independent scalar quantities. These six scalars arranged in an ordered 3×3 matrix forms the *stress tensor*,

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}.$$

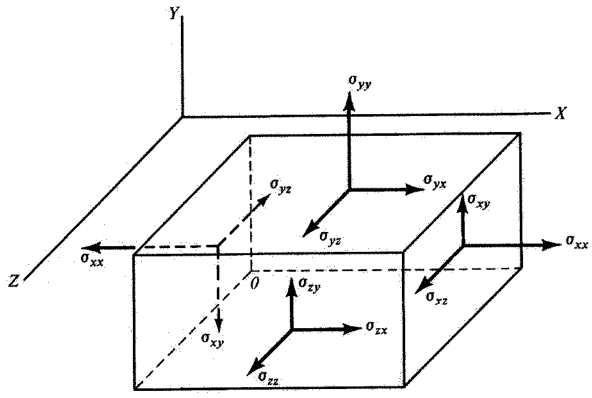


Fig. 3–2.2.

For example, the stress tensor for a cylinder with cross-sectional area A_0 in uniaxial tension from force $\mathbf{F}$ is

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \frac{F}{A_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ if the cylinder axis and $\mathbf{F}$ are both parallel to the x-axis,

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{F}{A_0} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ if the cylinder axis and $\mathbf{F}$ are both parallel to the y-axis,

and $$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{F}{A_0} \end{bmatrix}$$ if the cylinder axis and $\mathbf{F}$ are both parallel to the z-axis.

The sign convention for the stress elements is that a positive force on a positive face or a negative force on a negative face is positive. All others are negative. As a final example, a cube oriented so that its faces are perpendicular to the coordinate axes, with an area per face of A_0 has the following forces applied to it: force $\mathbf{F}_1$ applied to the positive x face in the positive x direction, force $\mathbf{F}_2$ applied to the positive y face in the positive y direction, and force $\mathbf{F}_3$ applied to the positive x face in the negative y direction. The necessary forces to keep the cube form moving are applied to the other faces. The resultant stress tensor is

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \frac{F_1}{A_0} & \frac{-F_3}{A_0} & 0 \\ \frac{-F_3}{A_0} & \frac{F_2}{A_0} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Stress on a plane.

It is often necessary to calculate the stress on an arbitrarily-oriented plane with normal $\hat{\mathbf{n}}$ inside a solid. A force balance on the tetrahedron formed by the intersection of the plane with the coordinate axes provides the needed results. We define a stress vector, $\mathbf{s}$, defined as limit of the net force acting on the plane, $\mathbf{F}$, per unit area as the area shrinks to zero. This vector can be decomposed into the *normal stress* on the plane (the force per unit area in the direction normal to the plane), $\mathbf{s}_n$, and the *shear stress* on the plane (the force per unit area in a direction lying in the plane), $\mathbf{s}_s$. It can also be decomposed into components in the three coordinate-axis directions, $\mathbf{s}_x$, $\mathbf{s}_y$, and $\mathbf{s}_z$.

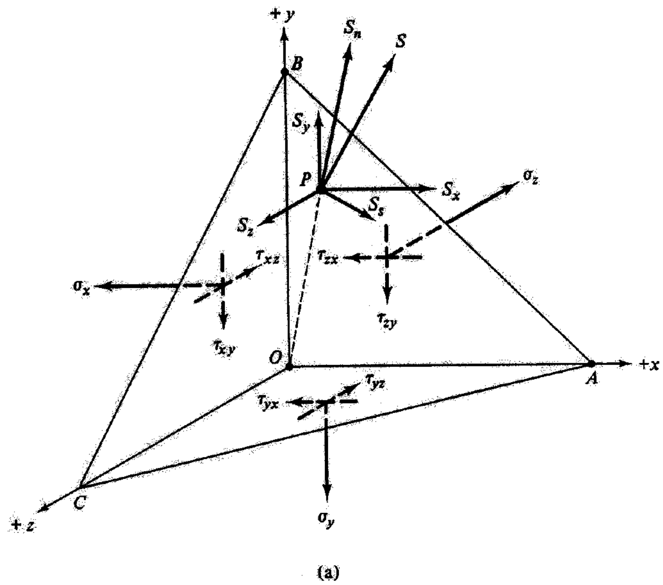
In other words, $\mathbf{s} = \mathbf{s}_x + \mathbf{s}_y + \mathbf{s}_z = s_x\hat{\mathbf{x}} + s_y\hat{\mathbf{y}} + s_z\hat{\mathbf{z}}$. If we define the direction cosines of $\hat{\mathbf{n}}$ as

$k = \hat{\mathbf{n}} \bullet \hat{\mathbf{x}} = \cos\alpha$, $l = \hat{\mathbf{n}} \bullet \hat{\mathbf{y}} = \cos\beta$, and $m = \hat{\mathbf{n}} \bullet \hat{\mathbf{z}} = \cos\gamma$,

then

$s_x = \sigma_{xx}k + \sigma_{xy}l + \sigma_{xz}m$, $s_y = \sigma_{xy}k + \sigma_{yy}l + \sigma_{yz}m$, and

$s_z = \sigma_{xz}k + \sigma_{yz}l + \sigma_{zz}m$,



or

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \sigma_{ij}\hat{\mathbf{n}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} k \\ l \\ m \end{bmatrix}.$$

We can then find the normal component of the stress, s_n, by

$$s_n = \mathbf{s} \bullet \hat{\mathbf{n}} = \begin{bmatrix} s_x & s_y & s_z \end{bmatrix} \begin{bmatrix} k \\ l \\ m \end{bmatrix}.$$

The shear component can be determined but requires a little more work.

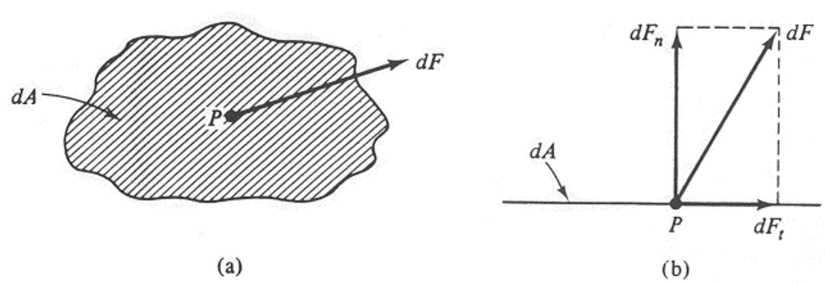


Figure 1-1 Forces acting on an elemental area showing total force (a) and resolved forces (b).

Coordinate transformations and stress invariants.

It is often useful to know the stress tensor in a coordinate system that has been rotated and/or translated with respect to the original coordinate system. We can transform the coordinates, $\begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$ into the coordinates, $\begin{bmatrix} x' & y' & z' \end{bmatrix}^{\mathrm{T}}$ by use of a transformation matrix, $\mathbf{Q}$, where $\mathbf{Q}$ has the property $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. In particular, a rotation about the z-axis through an angle θ is given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} .$$

To get all of the elements of the stress tensor in the new coordinate system,

$$\boldsymbol{\sigma}' = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}} .$$

The above relationship is often used to define a tensor of rank 2. Several properties of the stress tensor remain unchanged by a change in coordinates. These properties are called *invariants*. These invariants are closely related to important quantities. The *first invariant*, I_1, is the trace of the matrix,

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} .$$

The hydrostatic component of σ_{ij} (the part due to uniform pressure on all exterior surfaces of the solid) is equal to $I_1 / 3$.

The *second invariant*, I_2 is given by

$$I_2 = \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} .$$

The *third invariant* is

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 .$$

A common question in stress problem is: Is there a coordinate system for which all of the shear stresses disappear, and the remaining stresses are purely tensile or compressive? It turns out that there is. The resultant stresses are called the *principal stresses*, the planes on which they occur are the *principal planes*, and the directions of the resultant force components are the *principal directions* or *principal axes*. If we call the principal stresses λ_1, λ_2, and λ_3, then the problem appears as: Are there values of λ for which

$$\mathrm{Det}\begin{bmatrix} \sigma_{xx}-\lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\lambda \end{bmatrix} = 0\,?$$

The principal stresses are the eigenvalues and the principal directions are the eigenvectors. The eigenvalue problem can be rewritten in terms of the three invariants as $\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0$.

For any stress tensor, three real (but possibly not distinct) roots will result.

The Von Mises yielding criterion.

In a complex stress field it is not easy to determine if the stress has exceeded the yield stress in the body. Von Mises proposed the following criterion: Yielding occurs when the second invariant of the stress deviator exceeds some critical value.

The stress deviator is the stress tensor with the hydrostatic component removed, *i.e.*,

$$\sigma_{ij}^D = \sigma_{ij} - h\mathbf{I} = \begin{bmatrix} \sigma_{xx}-h & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-h & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-h \end{bmatrix} \text{ where } h = \frac{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}}{3}.$$

The second invariant of σ_{ij}^D is

$$I_2 = \frac{(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2)}{6},$$

which must be greater than some constant k^2 for yielding to occur. In terms of the yield stress, σ_y, the criterion is

$$\left[\frac{(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{xz}^2)}{2}\right]^{1/2} \geq \sigma_y$$

for yielding to occur.

The Strain Tensor.

Normal strain is the change in length in a given direction divided by the initial length in that direction. Shear strain is the complement of the angle between two initially perpendicular line segments. If you apply a force to a solid object you may end up simultaneously translating, rotating and deforming the object. The vector function which describes the difference between the initial position and the final position of each point in the object is

$$\mathbf{u}(x,y,z) = \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix}.$$

If we take the gradient of $\mathbf{u}$ we end up with a tensor

$$\nabla\mathbf{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}.$$

For small strains we find that $\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}} = 2\boldsymbol{\epsilon}$ where

$$\boldsymbol{\epsilon} = \epsilon_{kl} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}.$$

The diagonal terms are the normal strains in the x, y, and z directions respectively. The off-diagonal terms are equal to one-half of the engineering shear strain, *e.g.*, $\epsilon_{xy} = \gamma_{xy}/2$. In terms of $\nabla\mathbf{u}$,

$$\epsilon_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right],\ \epsilon_{xz} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right],\ \text{and}\ \epsilon_{yz} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right].$$

As was the case with stress $\epsilon_{yx} = \epsilon_{xy}$, $\epsilon_{yz} = \epsilon_{zy}$, and $\epsilon_{xz} = \epsilon_{zx}$. We can determine the strains in a rotated coordinate system in the same way as for stresses. We can transform the coordinates, $\begin{bmatrix} x & y & z \end{bmatrix}^{\mathrm{T}}$ into the coordinates, $\begin{bmatrix} x' & y' & z' \end{bmatrix}^{\mathrm{T}}$ by use of a transformation matrix, $\mathbf{Q}$, where $\mathbf{Q}$ has the property $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. To get all of the elements of the strain tensor in the new coordinate system, $[\boldsymbol{\epsilon}'] = \mathbf{Q}[\boldsymbol{\epsilon}]\mathbf{Q}^{\mathrm{T}}$.

Relationship between stress and strain.

Every member of ϵ_{kl} will cause a corresponding stress in σ_{ij}. The relationship can be written as $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$. Writing out the first term explicitly should suffice to explain the notation.

$$\begin{aligned}\sigma_{xx} = {} & C_{xxxx}\epsilon_{xx} + C_{xxxy}\epsilon_{xy} + C_{xxxz}\epsilon_{xz} \\ & + C_{xxyx}\epsilon_{yx} + C_{xxyy}\epsilon_{yy} + C_{xxyz}\epsilon_{yz} \\ & + C_{xxzx}\epsilon_{zx} + C_{xxzy}\epsilon_{zy} + C_{xxzz}\epsilon_{zz}\end{aligned}.$$

Fortunately only 21 of the 81 C_{ijkl}-terms are unique. To simplify the notation, the stress and strain tensors are rewritten as vectors. The simplified notation is known as *contracted notation*. First the off-diagonal strain terms are converted to engineering shear strains.

$$\begin{bmatrix} \epsilon_{xx} & 2\epsilon_{xy} & 2\epsilon_{xz} \\ 2\epsilon_{yx} & \epsilon_{yy} & 2\epsilon_{yz} \\ 2\epsilon_{zx} & 2\epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix}.$$

The resulting matrix is no longer a tensor because it doesn't follow the coordinate-transformation rules. Then the elements are renumbered.

$$
\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix}, \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_1 & \epsilon_6 & \epsilon_5 \\ \epsilon_6 & \epsilon_2 & \epsilon_4 \\ \epsilon_5 & \epsilon_4 & \epsilon_3 \end{bmatrix}.
$$

Then the matrices are written as vectors,

$$
\begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}, \begin{bmatrix} \epsilon_1 & \epsilon_6 & \epsilon_5 \\ \epsilon_6 & \epsilon_2 & \epsilon_4 \\ \epsilon_5 & \epsilon_4 & \epsilon_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}.
$$

Finally the relationships between the stress vector and the strain vector is expressed.

$$
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & Q_{45} & Q_{46} \\ Q_{15} & Q_{25} & Q_{35} & Q_{45} & Q_{55} & Q_{56} \\ Q_{16} & Q_{26} & Q_{36} & Q_{46} & Q_{56} & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}.
$$

The materials-property matrix with all of the Q's is known as the *stiffness matrix*. Unfortunately $\mathbf{Q}$ is used for both the stiffness matrix and the coordinate transformation matrix. Don't get them confused. The stiffness matrix is used when all of the strains are known and the values of the stresses are to be determined. In the more common case of the stresses being known and the strains to be determined, the inverse of the stiffness matrix, called the *compliance matrix,* $\mathbf{S}$, must be used.

The relationship between $\mathbf{Q}$ and $\mathbf{S}$ is that $\mathbf{S} = \mathbf{Q}^{-1}$. There are a number of simplified cases for the stiffness and compliance matrices.

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}.$$

Linear elastic isotropic materials:

The simplest materials are ones in which the properties do not vary with direction, or *linear elastic isotropic* materials. In a linear elastic isotropic material characterized by Young's modulus, E, Poisson's ratio, ν, and the shear modulus, $G = \dfrac{E}{2(1+\nu)}$, the following relationships between the stress tensor and the strain tensor hold:

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})],$$

$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})],$$

and

$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})],$$

also

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\sigma_{xy}}{G},$$

$$\gamma_{xz} = 2\epsilon_{xz} = \frac{\sigma_{xz}}{G},$$

and

$$\gamma_{yz} = 2\epsilon_{yz} = \frac{\sigma_{yz}}{G}$$

It is left to the reader (usually on an exam) to wade through the alphabet soup and determine the stiffness and compliance matrices for a linear elastic isotropic material.

Linear elastic orthotropic materials:

For an *orthotropic* material (one in which the properties in the y- and z-directions are the same but different in the x-direction, such as a composite material with the fibers all oriented in the x-direction) the stiffness matrix has the form

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix},$$

and the compliance matrix has the form

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}.$$

Since the properties in the y and z directions are equal, the two-dimensional x,y-case is often considered when determining materials properties. In this case the stiffness-matrix relationship becomes

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

and the compliance-matrix relationship becomes

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}.$$

First a uniaxial tensile test is performed in the x-direction. σ_{xx} is applied and ϵ_{xx}, and ϵ_{yy} are measured. As always in a uniaxial tensile test, $\sigma_{xx} = E_{cl}\epsilon_{xx}$, and $\nu_{lt} = -\frac{\epsilon_{yy}}{\epsilon_{xx}}$, where E_{cl} is the elastic modulus in the longitudinal or x direction, and ν_{lt} is Poisson's ratio for stress in the longitudinal direction and strain in the transverse direction. By comparison with the compliance matrix, it is seen that

$S_{11} = \frac{1}{E_{cl}}$, and $S_{12} = -\frac{\nu_{lt}}{E_{cl}}$.

Next a uniaxial tensile test is performed in the y-direction. σ_{yy} is applied and ϵ_{xx}, and ϵ_{yy} are measured. As before,

$\sigma_{yy} = E_{ct}\epsilon_{yy}$, and $\nu_{tl} = -\frac{\epsilon_{xx}}{\epsilon_{yy}}$, where E_{ct} is the elastic modulus in the transverse or y direction, and ν_{tl} is Poisson's ratio for stress in the transverse direction and strain in the longitudinal direction. By comparison with the compliance matrix, it is seen that

$S_{22} = \frac{1}{E_{ct}}$, and $S_{12} = -\frac{\nu_{tl}}{E_{ct}}$.

A final test with pure shear gives the relationship

$S_{66} = \frac{1}{G}$.

The interested reader should be able to determine the relationship between ν_{lt} and

ν_{tl} from the relationships $S_{12} = -\frac{\nu_{lt}}{E_{cl}}$ and $S_{12} = -\frac{\nu_{tl}}{E_{ct}}$.

The structure of $\mathbf{Q}$ is not nearly so simply related to E_{cl}, E_{ct}, ν_{lt}, and ν_{tl} as is $\mathbf{S}$, but, as before, it can be determined from $\mathbf{SQ} = \mathbf{I}$, or $\mathbf{S} = \mathbf{Q}^{-1}$.