Review of Vectors and Dot Products

Nomenclature: Vectors and matrices are represented by **boldfaced** letters. Scalars (such as the magnitude of a vector) are represented by *italicized* letters.

The unit vectors in the x, y, z directions are known as either $\mathbf{i}, \mathbf{j}, \mathbf{k}$, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, or

$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

The magnitude of a vector, $\mathbf{a}$, can be written either as $|\mathbf{a}|$ or a and is evaluated by

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

If two vectors have an angle of θ between them, the dot product is defined as:

$\mathbf{a} \bullet \mathbf{b} = ab\cos\theta$. It can also be found by $\mathbf{a} \bullet \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$.

The direction cosines of a vector $\mathbf{a}$ are k, l, m and are given by:

$$k = \frac{\mathbf{a} \bullet \hat{\mathbf{x}}}{a},\ l = \frac{\mathbf{a} \bullet \hat{\mathbf{y}}}{a},\ m = \frac{\mathbf{a} \bullet \hat{\mathbf{z}}}{a}.$$

To find the component of $\mathbf{a}$ in the direction of $\mathbf{b}$, or the projection of $\mathbf{a}$ in the direction of $\mathbf{b}$, use

$$a_b = \frac{\mathbf{a} \bullet \mathbf{b}}{b} = a\cos\theta.$$

Example: A force of magnitude 10 is applied in the [121] direction. What is the component of this force in the $[2\bar{3}5]$?

First method: Let $\mathbf{a} = 1\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 1\hat{\mathbf{z}}$, and $\mathbf{b} = 2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + 5\hat{\mathbf{z}}$. Then

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{ab} = \frac{(1 \cdot 2) + (2 \cdot -3) + (1 \cdot 5)}{\sqrt{1 + 4 + 1}\sqrt{4 + 9 + 25}} = \frac{1}{\sqrt{6}\sqrt{38}} = 0.06622662 \text{ and}$$

$$F_b = 10\frac{1}{\sqrt{6}\sqrt{38}} = 0.66226618$$

Second method: Find the two normal vectors, $\mathbf{n}_a$, and $\mathbf{n}_b$.

$$\mathbf{n}_a = \frac{\mathbf{a}}{a} = \frac{1}{\sqrt{6}}\hat{\mathbf{x}} + \frac{2}{\sqrt{6}}\hat{\mathbf{y}} + \frac{1}{\sqrt{6}}\hat{\mathbf{z}},\ \mathbf{n}_b = \frac{\mathbf{b}}{b} = \frac{2}{\sqrt{38}}\hat{\mathbf{x}} - \frac{3}{\sqrt{38}}\hat{\mathbf{y}} + \frac{5}{\sqrt{38}}\hat{\mathbf{z}}.$$

Then

$$\mathbf{F} = F\mathbf{n}_a = \frac{10}{\sqrt{6}}\hat{\mathbf{x}} + \frac{20}{\sqrt{6}}\hat{\mathbf{y}} + \frac{10}{\sqrt{6}}\hat{\mathbf{z}} \text{ and } F_b = \mathbf{F} \bullet \mathbf{n}_b = \frac{20 - 60 + 50}{\sqrt{6}\sqrt{38}} = 0.66226618\,.$$

Areas: An area in a plane can be represented by a vector normal to the plane with a magnitude equal to the area on the plane: $\mathbf{A} = A\mathbf{n}$. We can find the projection of that area into a different plane with unit normal $\mathbf{n}_b$ by using the dot product: $A_b = \mathbf{A} \bullet \mathbf{n}_b$.